The Firm in the 2-pd OLG Model

In our OLG model the firm has a neoclassical production function,
\[ Y = F(K, L), \]
with positive marginal products
\[ F_K(K, L) = \frac{\partial F}{\partial K} > 0, \quad F_L(K, L) = \frac{\partial F}{\partial L} > 0, \quad \forall K, L > 0. \]
It is also subject to diminishing marginal products,
\[ F_{KK}(K, L) < 0, \quad F_{LL}(K, L) < 0, \quad \forall K, L > 0. \]
It is a constant returns to scale (CRS) production function, hence,
\[ F(\lambda K, \lambda L) = \lambda F(K, L) \quad \forall \lambda, K, L > 0. \]
Since \( F(\cdot) \) is homogeneous of degree 1 (HD1),
\[ F_K(\cdot), \quad F_L(\cdot) \text{ are HD0,} \]
and the following production exhaustion theorem holds,
\[ F(K, L) = K.F_K(K, L) + L.F_L(K, L). \]
In a competitive equilibrium, factors are paid their marginal products. So, by the production exhaustion theorem, all the produced output is exhausted once the factors are paid off.

\textit{Intensive Form of Production Function}

Under CRS, output per worker (more precisely, output per number of work hours) depends on the \((K/L)\) ratio.
\[ F(\lambda K, \lambda L) = \lambda F(K, L) \quad \forall \lambda, K, L > 0. \]
\[ \Rightarrow \quad F\left(\frac{K}{L}, 1\right) = \frac{F(K, L)}{L}, \quad \text{if } \lambda = \frac{1}{L} \]
\[ \Rightarrow \quad F(k, 1) = y, \quad \text{if } y = \frac{F}{L}, \quad k = \frac{K}{L} \]
\[ \Rightarrow \quad f(k) = y. \]
\( F(k, 1) \equiv f(k) \) is the \textit{intensive form} of production function, which indicates output that one worker produces with her share of the capital. Some of the properties of this function are:
\[ f(0) = 0, \]
\[ f'(k) > 0, \]
\[ f''(k) < 0, \]
with the Inada conditions,
\[ \lim_{k \to 0} f'(k) = \infty, \]
\[ \lim_{k \to \infty} f'(k) = 0. \]
Factor Prices Under CRS and Perfect Competition

Since \( F(K,L) \) is HD1 and \((F_{KK}, F_{LL})\) are HD0,

\[
F_K(K,L) = F_K\left(\frac{K}{L}, \frac{L}{L}\right) = F(k,1) = f'(k).
\]

Now, using Production Exhaustion Theorem,

\[
y = \frac{F(K,L)}{L} = \frac{K}{L}.F_K(K,L) + \frac{L}{L}.F_L(K,L)
\]

\[
\Rightarrow f(k) = k.f'(k) + F_L(K,L)
\]

\[
\Rightarrow F_L(K,L) = f(k) - k.f'(k)
\]

The above hold under CRS. Now, under perfect competition,

\[
w = F_L(K,L) = f(k) - k.f'(k) = w(k),
\]

\[
r = F_K(K,L) = f'(k) = r(k),
\]

where, \( w \) and \( r \) denote wage and rent, respectively. Also,

\[
\frac{\partial w(k)}{\partial k} = f'(k) - k.f''(k) - f'(k) = -k.f''(k) > 0,
\]

\[
\frac{\partial r(k)}{\partial k} = f''(k) < 0.
\]