Time Value of Money
Time Value of Money

- Which would you rather receive as a sign-in bonus for your new job?
  1. $15,000 cash upon signing the contract
  2. $15,000 cash at the end of first year of employment
  3. $20,000 cash at the end of the first year of employment

  - Cash in hand seems to be no brainer, but!

- How would be determine whether (3) is better or worse than (1)?
The One-Period Case

• If you were to invest $10,000 at 5-percent interest for one year, your investment would grow to $10,500.

  $500 would be interest ($10,000 \times .05)
  $10,000 is the principal repayment ($10,000 \times 1)
  $10,500 is the total due. It can be calculated as:

  $10,500 = $10,000 \times (1.05)$

• The total amount due at the end of the investment is call the Future Value (FV).
Future Value

- In the one-period case, the formula for $FV$ can be written as:

$$FV = C_0 \times (1 + r)$$

Where $C_0$ is cash flow today (time zero), and $r$ is the appropriate interest rate.
Present Value

- If you were to be promised $10,000 due in one year when interest rates are 5-percent, your investment would be worth $9,523.81 in today’s dollars.

\[ $9,523.81 = \frac{\$10,000}{1.05} \]

- The amount that a borrower would need to set aside today to be able to meet the promised payment of $10,000 in one year is called the **Present Value (PV)**.

  - Note that $10,000 = $9,523.81 \times (1.05)$. 
Present Value

- In the one-period case, the formula for \( PV \) can be written as:

\[
P V = \frac{C_1}{1 + r}
\]

- Where \( C_1 \) is cash flow at date 1, and
- \( r \) is the appropriate interest rate.
Net Present Value

- The Net Present Value (NPV) of an investment is the present value of the expected cash flows, less the cost of the investment.

- Suppose an investment that promises to pay $10,000 in one year is offered for sale for $9,500. Your interest rate is 5%. Should you buy?
Net Present Value

\[ NPV = -9,500 + \frac{10,000}{1.05} \]

\[ NPV = -9,500 + 9,523.81 \]

\[ NPV = 23.81 \]

The present value of the cash inflow is greater than the cost. In other words, the Net Present Value is positive, so the investment should be purchased.
In the one-period case, the formula for \( NPV \) can be written as:

\[
NPV = - \text{Cost} + \text{PV}
\]

If we had not undertaken the positive \( NPV \) project considered on the last slide, and instead invested our $9,500 elsewhere at 5 percent, our \( FV \) would be less than the $10,000 the investment promised, and we would be worse off in \( FV \) terms:

\[
9,500 \times (1.05) = 9,975 < 10,000
\]
The Multiperiod Case

The general formula for the future value of an investment over many periods can be written as:

\[ FV = C_0 \times (1 + r)^T \]

Where

- \( C_0 \) is cash flow at date zero,
- \( r \) is the appropriate interest rate, and
- \( T \) is the number of periods over which the cash is invested.
The general formula for the present value of an investment over many periods can be written as:

\[ PV = \frac{C_T}{(1 - r)^T} \]

Where
- \( C_T \) is cash flow at date \( T \),
- \( r \) is the appropriate interest rate, and
- \( T \) is the number of periods over which the cash is invested.
Future Value

• Suppose a stock currently pays a dividend of $1.10, which is expected to grow at 40% per year for the next five years.

• What will the dividend be in five years?

\[ FV = C_0 \times (1 + r)^T \]

\[ $5.92 = $1.10 \times (1.40)^5 \]
Future Value and Compounding

• Notice that the dividend in year five, $5.92, is considerably higher than the sum of the original dividend plus five increases of 40-percent on the original $1.10 dividend:

\[
5.92 > 1.10 + 5 \times [1.10 \times 0.40] = 3.30
\]

This is due to compounding.
Future Value and Compounding

$1.10 \times (1.40)^5$

$1.10 \times (1.40)^4$

$1.10 \times (1.40)^3$

$1.10 \times (1.40)^2$

$1.10 \times (1.40)$

$1.10$ $1.54$ $2.16$ $3.02$ $4.23$ $5.92$

$0$ $1$ $2$ $3$ $4$ $5$
Present Value and Discounting

- How much would an investor have to set aside today in order to have $20,000 five years from now if the current rate is 15%?

\[ PV = \frac{FV}{(1 + r)^n} \]

\[ PV = \frac{20,000}{(1.15)^5} \approx 9,943.53 \]
How Long is the Wait?

If we deposit $5,000 today in an account paying 10%, how long does it take to grow to $10,000?

\[ FV = C_0 \times (1+r)^T \]
\[ \$10,000 = \$5,000 \times (1.10)^T \]
\[ (1.10)^T = \frac{\$10,000}{\$5,000} = 2 \]
\[ \ln((1.10)^T) = \ln(2) \]
\[ T = \frac{\ln(2)}{\ln(1.10)} = \frac{0.6931}{0.0953} = 7.27 \text{ years} \]
What Rate Is Enough?

Assume the total cost of a college education will be $50,000 when your child enters college in 12 years. You have $5,000 to invest today. What rate of interest must you earn on your investment to cover the cost of your child’s education?

\[ FV = C_0 \times (1 + r)^T \]

\[ $50,000 = $5,000 \times (1 + r)^{12} \]

\[ (1 + r)^{12} = \frac{$50,000}{$5,000} = 10 \]

\[ (1 + r) = 10^{1/12} \]

\[ r = 10^{1/12} - 1 = 1.2115 - 1 = .2115 \]

About 21.15%. 

Business Finance: FIN 5013  Prof. Ali Nejadmalayeri
Calculator Keys

• Texas Instruments BA-II Plus
  – FV = future value
  – PV = present value
  – I/Y = periodic interest rate
    • P/Y must equal 1 for the I/Y to be the periodic rate
    • Interest is entered as a percent, not a decimal
  – N = number of periods
  – Remember to clear the registers (CLR TVM) after each problem
  – Other calculators are similar in format
Multiple Cash Flows

- Consider an investment that pays $200 one year from now, with cash flows increasing by $200 per year through year 4. If the interest rate is 12%, what is the present value of this stream of cash flows?

- If the issuer offers this investment for $1,500, should you purchase it?
Multiple Cash Flows

0 1 2 3 4

200 400 600 800

178.57 318.88 427.07 508.41 1,432.93

Present Value < Cost → Do Not Purchase
Valuing “Lumpy” Cash Flows

First, set your calculator to 1 payment per year.

Then, use the cash flow menu:

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<th>CF0</th>
<th>CF3</th>
<th>CF4</th>
</tr>
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<td>800</td>
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<table>
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<table>
<thead>
<tr>
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<td>1</td>
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<tr>
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NPV = $1,432.93
Compounding Periods

Compounding an investment $m$ times a year for $T$ years provides for future value of wealth:

$$FV = C_0 \times \left(1 + \frac{r}{m}\right)^{m \times T}$$
Compounding Periods

- For example, if you invest $50 for 3 years at 12% compounded semi-annually, your investment will grow to

\[ FV = \$50 \times \left(1 + \frac{0.12}{2}\right)^{2 \times 3} = \$50 \times (1.06)^6 = \$70.93 \]
A reasonable question to ask in the above example is “what is the effective annual rate of interest on that investment?”

\[
FV = 50 \times (1 + \frac{.12}{2})^{2 \times 3} = 50 \times (1.06)^6 = 70.93
\]

The Effective Annual Rate (EAR) of interest is the annual rate that would give us the same end-of-investment wealth after 3 years:

\[
50 \times (1 + EAR)^3 = 70.93
\]
Effective Annual Rates of Interest

\[ FV = \$50 \times (1 + EAR)^3 = \$70.93 \]

\[ (1 + EAR)^3 = \frac{\$70.93}{\$50} \]

\[ EAR = \left( \frac{\$70.93}{\$50} \right)^{1/3} - 1 = .1236 \]

So, investing at 12.36% compounded annually is the same as investing at 12% compounded semi-annually.
Effective Annual Rates of Interest

• Find the Effective Annual Rate (EAR) of an 18% APR loan that is compounded monthly.

• What we have is a loan with a monthly interest rate rate of 1½%.

• This is equivalent to a loan with an annual interest rate of 19.56%.

\[
\left(1+\frac{r}{m}\right)^{n\times m} = \left(1+\frac{0.18}{12}\right)^{12} = (1.015)^{12} = 1.1956
\]
Texas Instruments BAII Plus

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<th>description:</th>
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<tr>
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Continuous Compounding

• The general formula for the future value of an investment compounded continuously over many periods can be written as:

\[ FV = C_0 \times e^{rT} \]

Where

- \( C_0 \) is cash flow at date 0,
- \( r \) is the stated annual interest rate,
- \( T \) is the number of years, and
- \( e \) is a transcendental number approximately equal to 2.718. \( e^x \) is a key on your calculator.
Classical Cash Flow Schemes

- **Perpetuity**
  - A constant stream of cash flows that lasts forever

- **Growing perpetuity**
  - A stream of cash flows that grows at a constant rate forever

- **Annuity**
  - A stream of constant cash flows that lasts for a fixed number of periods

- **Growing annuity**
  - A stream of cash flows that grows at a constant rate for a fixed number of periods
Perpetuity

A constant stream of cash flows that lasts forever

\[ PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \]

\[ PV = \frac{C}{r} \]
What is the value of a British consol that promises to pay £15 every year for ever? The interest rate is 10-percent.

\[ PV = \frac{\£15}{.10} = \£150 \]
Growing Perpetuity

A growing stream of cash flows that lasts forever

\[ PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \ldots \]

\[ PV = \frac{C}{r - g} \]
Growing Perpetuity: Example

The expected dividend next year is $1.30, and dividends are expected to grow at 5% forever.

If the discount rate is 10%, what is the value of this promised dividend stream?

\[
PV = \frac{1.30}{0.10 - 0.05} = $26.00
\]
Annuity

A constant stream of cash flows with a fixed maturity

\[ PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^T} \]

\[ PV = \frac{C}{r} \left[ 1 - \frac{1}{(1+r)^T} \right] \]
Annuity: Example

If you can afford a $400 monthly car payment, how much car can you afford if interest rates are 7% on 36-month loans?

\[
PV = \frac{400}{.07/12} \left[ 1 - \frac{1}{(1+.07/12)^{36}} \right] = 12,954.59
\]
What is the present value of a four-year annuity of $100 per year that makes its first payment two years from today if the discount rate is 9%?

\[
P V_1 = \sum_{t=1}^{4} \frac{100}{(1.09)^t} = \frac{100}{(1.09)^1} + \frac{100}{(1.09)^2} + \frac{100}{(1.09)^3} + \frac{100}{(1.09)^4} = 323.97
\]

\[
PV_0 = \frac{327.97}{1.09} = 297.22
\]
Growing Annuity

A growing stream of cash flows with a fixed maturity

\[ PV = \frac{C}{(1+r)} + \frac{C \times (1+g)}{(1+r)^2} + \cdots + \frac{C \times (1+g)^{T-1}}{(1+r)^T} \]

\[ PV = \frac{C}{r-g} \left[ 1 - \left( \frac{1+g}{1+r} \right)^T \right] \]
Growing Annuity: Example

A defined-benefit retirement plan offers to pay $20,000 per year for 40 years and increase the annual payment by 3% each year. What is the present value at retirement if the discount rate is 10%?

\[
PV = \frac{\$20,000}{.10 - .03} \left[ 1 - \left( \frac{1.03}{1.10} \right)^{40} \right] = \$265,121.57
\]
Growing Annuity: Example

You are evaluating an income generating property. Net rent is received at the end of each year. The first year's rent is expected to be $8,500, and rent is expected to increase 7% each year. What is the present value of the estimated income stream over the first 5 years if the discount rate is 12%?

\[
egin{align*}
8,500 	imes (1.07)^4 &= 11,141.77 \\
8,500 	imes (1.07)^3 &= 10,412.87 \\
8,500 	imes (1.07)^2 &= 9,731.65 \\
8,500 	imes (1.07) &= 9,095 \\
8,500 &= 8,500
\end{align*}
\]

$34,706.26
What Is a Firm Worth?

- Conceptually, a firm should be worth the present value of the firm’s cash flows.
- The tricky part is determining the size, timing, and risk of those cash flows.