No-Arbitrage and Equilibrium Pricing in Complete Markets:

Imagine a finite state space with $s \in \{1, \ldots, S\}$ where there exist $n$ traded assets with a price vector $p = \{p_1, \ldots, p_n\}$. A portfolio $w = \{w_1, \ldots, w_n\}$ is then an arbitrage portfolio if: $pw \leq 0$ and $\Pi w > 0$, where $\Pi$ is the “Arrow-Debreu” tableau of prices.

The “Arrow-Debreu” price is the price of the “Arrow-Debreu” security (also known as “state security”, “state-contingent claim”, and “pure security”): a security that pays off one unit of numeraire in one and only one future state of nature.

Markets are called complete if payoff matrix for all assets has a rank $S$. In these markets, then “market equivalent theorem” holds.
Market Equivalent Theorem:

Consider an economy with as many different securities as states (i.e., complete market). The equilibrium portfolio choice of each investor and the equilibrium security prices will be the same in that economy as in an otherwise identical economy, but where the original securities are replaced with a full set of state-securities. For instance, a risk-less asset’s return is given as $R_f = (1'\Pi)^{-1} = \sum_{i=1}^{S} \pi_i$, where 1 is a vector of ones and $\Pi$ is the “Arrow-Debreu” tableau of prices.
The Fundamental Theorem of Finance:

The following are equivalent:

1. No Arbitrage

2. The existence of a positive linear pricing rule that prices all assets

3. The existence of a (finite) optimal demand for some agent who prefers more to less

Proof: Dybvig and Ross (1987). The upshot of their proof is that in complete markets with no arbitrage then there exists a unique linear pricing rule, $q$, such that $p = q\Pi$. The linear pricing rule is then given by $q = p\Pi^{-1}$. 
**Definition:** Martingale or risk-neutral probabilities are a set of probabilities, $\pi^*$, such that for any asset with a future payoff vector $X = \{x_1, \ldots, x_S\}$, the value is given by:

$$p(X) = \frac{1}{1 + R_f} E^*[X] = \sum_{s=1}^{S} \pi_s^* x_s$$

**Definition:** A state price density or pricing kernel is a vector, $M = \{m_1, \ldots, m_S\}$, such that for any asset with payoff vector $X = \{x_1, \ldots, x_S\}$, the value is given by:

$$p(X) = E[MX] = \sum_{s=1}^{S} \pi_s m_s x_s$$
Representation Theorem:

The following are equivalent:

1. There exists a linear pricing rule.

2. The martingale property: the existence of risk neutral probabilities and an associated riskless rate.

3. There exists a positive state price density or pricing kernel.

Proof: Ross (2004). Let’s assume a positive pricing vector $q \gg 0$. For any asset with pay-off vector $X = \{x_1, ..., x_S\}$, the value is given by $p(X) = qX$. The value of riskless asset is defined as $p(1) = q1 = \sum_{i=1}^{S} q_i$. 
Results of Representation Theorem:

First, we can show that risk-neutral probabilities and linear pricing vector are linked as following:

\[ \pi_i^* = \frac{q_i}{\sum_{i=1}^{S} q_i} > 0 \]

This means that risk-neutral probabilities and an associated riskless rate define state price vector as:

\[ q_i^* = \frac{1}{1 + R \pi_i^*} > 0 \]

Second, we can show that the positive price density (i.e., discount factor) is given by:

\[ m_i = \frac{q_i}{\pi_i} > 0 \]
General Asset Pricing Framework:

Assuming one-period ahead stochastic payoffs, $X_{i,t+1}$, and stochastic pricing kernel, $M_{t+1}$, for asset $i$th at time $t$, then the general pricing equation is given by:

$$p_{i,t} = E_t[M_{t+1}X_{i,t+1}]$$

Now recall Fisher’s economy, where representative agent $k$ solves an exchange problem as follows

$$\max_{\xi} U(C_{i,t}) + \beta E_t[U(C_{i,t+1})] \quad s.t. \quad W_{i,t} = C_{i,t} + \xi p_{i,t}$$

$$W_{i,t+1} = -C_{i,t+1} + \xi X_{i,t}$$
Stochastic Discount Factor and Marginal Utility of Consumption:

The first-order conditions (or Euler equation) describing investor behavior yields:

\[ U'(C_{i,t})p_{i,t} = \beta \mathbb{E}_t[U'(C_{i,t+1})X_{i,t+1}] \]

The intuition being that the marginal forgone utility of purchasing a share of asset \( i \) at time \( t \) should equate to the expected marginal utility of payoffs of asset \( i \) at time \( t + 1 \).

The above implies that “stochastic discount factor” or “pricing kernel” is equal to “intertemporal marginal rate of substitution”, or:

\[ M_{t+1} = \frac{\beta U'(C_{i,t+1})}{U'(C_{i,t})} \]
The Real Interest Rate:

Previous analysis yields that for any asset $k$:

$$1 = E_t[M_{t+1}(1 + R_{i,t+1})]$$

For a short-term riskless asset $f$ with a payoff of one tomorrow, then:

$$E_t[M_{t+1}] = p_{f,t} = \frac{1}{1 + R_{i,t+1}}$$

The risk-premium for any asset $i$ then is:

$$E_t[R_{i,t+1} - R_{f,t+1}] = \frac{-Cov_t(M_{t+1}, R_{i,t+1} - R_{f,t+1})}{E_t[M_{t+1}]}$$

and Sharpe ratio of asset $i$ and SDF are linked as following:

$$\frac{\sigma_t(M_{t+1})}{E_t[M_{t+1}]} \geq \frac{E_t[R_{i,t+1} - R_{f,t+1}]}{\sigma_t(R_{i,t+1} - R_{f,t+1})}$$
Equity Premium Puzzle:

- Surprisingly large equity return volatility (i.e., large equity Sharpe ratio) as compared to volatility of aggregate consumption

- Mehra and Prescott (1985) noted the extremely large risk aversion needed; coined “the equity premium puzzle”

- there are certain econometric issues:
  - mean returns are hard to estimate because the length of calendar time not number of observation matters, i.e., annual better than monthly, monthly better than daily, ...
  - selection bias due to abnormally high growth period is U.S.
  - Catastrophic risks are priced
Predictability of Aggregate Stock Returns:


- concerns: longer horizon more predictable, overlapping observation cause econometric issues, many determinants are persistent and highly correlated with returns

- state-of-art econometrics uses log-normal joint distributions for both SDF and returns

\[ E_t[r_{i,t+1} - r_{f,t+1}] + 0.5\sigma_i^2 = -\text{Cov}_t(m_{t+1}, r_{i,t+1}) \]
Government Bonds Returns:

- Treasury yield curve is upward sloped but highly convex; yield spread at 3-month, 1-year, and 2-year are 33 bps, 77 bps, and 96 bps and then remain unchanged.

- Hansen and Jagannathan (1991) note that steep yield curve implies a high volatility of SDF (do simple forward rate computation and show that this true!)

- Changes in Treasury term spread forecasts excess Treasury bond returns. This contradict pure expectation hypothesis that contends excess bond returns are constant. This predictive power of term spread is particularly large when there is greater seasonality and cyclical variations.
Factor Structure of SDF:

If all investors are single-period, mean-variance optimizers, then the market portfolio is mean-variance efficient, and there exists a beta (linear) pricing between all assets and the market portfolio. In more general sense, the SDF is then a linear combination of $K$ factors:

\[ M_{t+1} = a_t + \sum_{k=1}^{K} b_{k,t} f_{k,t+1} \]

The negative covariance of any excess return and the SDF is then:

\[ -\text{Cov}_t(M_{t+1}, R_{i,t+1} - R_{f,t+1}) = \sum_{k=1}^{K} b_{k,t} \sigma_{i,k,t+1} \]

\[ = \sum_{k=1}^{K} (b_{k,t} \sigma_{k,t+1}^2) \left( \frac{\sigma_{i,k,t+1}}{\sigma_{k,t+1}^2} \right) = \sum_{k=1}^{K} \lambda_{k,t} \beta_{i,k,t+1} \]
Cross-Section of Stock Returns:

- Early evidence [Black, Jensen, and Scholes (1972) and Fama and MacBeth (1973)] supported Sharpe-Litner CAPM pricing model under borrowing/lending constraints.

- Banz (1991) reported the size effect: small firms have higher excess return than large ones [Fama and French (1992, 1993, 1995)].

- Basu (1983) and others show that value stocks have higher excess returns as well [Fama and French (1992, 1993, 1995)].

- Jagadeesh and Titman (1993) show that there is momentum effect, past 12 month of high returns leads to better future return
Sources of Cross-Sectional Patterns:

- spurious results due to data snooping; small firm effect reversed in 15 year after found

- unknown true market index; labor income and human capital is not reflect in indexes

- difference between conditional and unconditional asset pricing models; case of distress risk

- mistakes not rather mispriced risk; patterns are found then exploited; investors over-emphasize past performance of firms
Prices, Returns, & Cash Flows:

With assets that last more than one-period, the pricing equation is general discounted dividend model:

\[ p_t = E_t \left[ \sum_{\tau=t+1}^{\infty} \frac{D_{t+\tau}}{(1 + R)^\tau} \right] \]

Campbell and Shiller (1988) extended the above to allow for log-linear dividends and time-varying discount rate:

\[ p_t - d_t = \frac{k}{1 - \rho} + E_t \left[ \sum_{\tau=0}^{\infty} \rho^\tau \left( \Delta d_{t+1+\tau} - r_{t+1+\tau} \right) \right] \]

Campbell’s (1991) decomposition:

\[ r_{t+1} - E_t[r_{t+1}] = [E_{t+1} - E_t] \left[ \sum_{\tau=0}^{\infty} \rho^\tau \Delta d_{t+1+\tau} \right] \]

\[ + [E_{t+1} - E_t] \left[ \sum_{\tau=0}^{\infty} \rho^\tau r_{t+1+\tau} \right] \]
Consumption & Portfolio Choice:

Imagine investor, $k$, who lives off financial wealth and consumes all wealth tomorrow, then the payoff $X_{k,t+1}^*$ on an optimal portfolio will be equal to terminal consumption $C_{k,t+1}$ and:

$$M_{t+1} = \theta_t U'_k(X_{k,t+1}^*) \Rightarrow X_{k,t+1}^* = U_k^{-1}((\theta_t^{-1} M_{t+1})$$

Single period examples: 1. Markowitz (1952) assumes quadratic utility hence mean-variance analysis 2. CAPM is the case of $M_{t+1}$ which is linear in the return on the market

Multi-period case: Samuelson (1969) shows a case where 1) investor has no labor income and 2) von Neumann-Morgenstern utility then sufficient conditions for myopic portfolio choice are either I) log utility or II) both power utility and IID returns.
Equilibrium Models of Representative Agent:

Lucas (1978) shows that equilibrium SDF can be obtained from exogenous consumption under some utility assumption. Three puzzles are found:

**Equity Risk Premium Puzzle** innovations in consumption are too small to cause observed large equity premium

**Stock Market Volatility Puzzle** variations of expected future consumption growth offsets variations of dividend growth and stock returns, leaving no room for observed large equity volatility

**Risk-free Rate Puzzle** historical upward drift in consumption implies desire to borrow from future, leading to abnormally high real risk-free rates
Habit Formation Models:

Sunderasan (1989) and Constantinides (1990) argue for a habit formation, a positive effect of today’s consumption on tomorrow’s marginal utility of consumption.

- Utility, $U(C_t, Y_t)$, can be power function of ratio $C_t/Y_t$, or power function of difference, $C_t - Y_t$.

- Habit can be “internal” – depends on agent’s own consumption – or “external” – depends on aggregate consumption (i.e., “catching up with the Jones”).

- Speed with which habit reacts to individual or aggregate consumption (depends on one lag vs. gradual change)
External Habit Formation Models:

Since an increase in previous consumption increases habit, the agent wants to increase today’s consumption by borrowing from future making interest rates instable and too high. Campbell and Cochrane (1999) propose a model of external habit where interest rates are stable and stocks can have high volatility. In their construct, the log of surplus $S_t \equiv (C_t - Y_t)/C_t$ can have a AR(1) process:

$$s_{t+1} = (1 - \varphi)\bar{s} + \varphi s_t + \lambda s_t \epsilon_{c,t+1}$$

In this setting, the SDF is given as:

$$M_{t+1} = \delta \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma}$$
Heterogenous Agents Models:

Two types of agents: 1) constrained who only consume their labor income and 2) unconstrained. Evidence suggest that consumption of stockholders is more volatile. Heaton and Lucas (1999) and Vissing-Jørgensen (1997) build general equilibrium models with a fraction of population in stock market. Constantinides and Duffie (1996) model says:

Individuals $k$ have different consumption levels $C_{k,t}$. The cross sectional distribution is lognormal and changes from $t$ to $t+1$ are uncorrelated. All investor have the same power utility. In this setup, then there is valid SDF, $M^{*}_{t+1}$, from intertemporal marginal rate of substitution and an invalid SDF, $M^{RA}_{t+1}$, from marginal utility of cross sectional average consumption. Then:

$$m^{*}_{t+1} - m^{RA}_{t+1} = \frac{\gamma(1-\gamma)}{2} Var^{*}_{t+1} \Delta c_{k,t+1}$$
Behavioral Models:

**Limits to Arbitrage** limited ability to absorb demands of noise traders, this could easily due to risk when utility-maximizing agents have to buy more risky assets from noise traders

**Prospect Theory** based on experimental works of Kahneman and Tversky (1979) which showed people 1) judge outcomes based on an initial reference point, and 2) have a kinked utility function; concave on gains and convex on losses.

**Irrational Expectation** In absence of arbitrage, there exists a unique state price vector which is product of subjective state probabilities and ratio of marginal utilities. Are subjective prob. equal to objective ones?