Importance of Interest Rates

- Economic decisions of the household
- Economic decisions of firms and businesses
- Overall performance of the economy

Overview

- Different people mean different things by “interest rate”: simple interest, yield to maturity, rate of return, all different concepts.
- Yield to Maturity: Economist’s view of interest rates
- Measuring Interest rates on different debt instruments
- Rate of return often does not equal the interest rate
- The Issue of inflation: Real versus nominal interest rates
Simple Interest

- Lender provides borrower with Principal
- Borrower pays back to lender at a maturity date
- Borrower also makes additional payment for interest along with the repayment

Example
Simple interest = 10% = 0.10
Loan amount = $100
Repayment = principle + interest
= 100 + (100 × 0.10) . . . . . . . . . . (*)
= 100 + 10 = 110

Note, we can rewrite (*),
100 + (100 × 0.10) = 100 × (1 + 0.10)

Concept of Present Value (PV)

A sequence of yearly simple interest (i=10%) starting with $100

Year 1: 100 × (1 + 0.10) = 110
Year 2: 110 + (110 × 0.10)
= 110 × (1 + 0.10)
= [100 × (1 + 0.10)] × (1 + 0.10)
    because, 110 = 100 × (1 + 0.10)
= 100 × (1 + 0.10)^2
= 121

Year 3: 121 + (121 × 0.10)
= 121 × (1 + 0.10)
= [100 × (1 + 0.10)^2] × (1 + 0.10)
    because, 121 = 100 × (1 + 0.10)^2
= 100 × (1 + 0.10)^3
= 133
**Concept of Present Value (PV)**

$100$ today yields $100 \times (1 + 0.10)^n$ after $n$ years

\[ \text{PV of } $100 \text{ n years from now} = \frac{100}{(1+i)^n} \]

Discount factor = $\frac{1}{(1+i)^n}$

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**Yield to Maturity (YTM)**

Economist’s view of interest rate = yield to maturity

YTM is the interest rate that makes, 

\[ \text{today’s value} = \text{present value of all future payments} \]
### YTM: Simple Loan

**Example:** Principle = $100, simple interest = 10%

Equation for YTM is,

Today’s value = present value of all future payments

\[
100 = \frac{110}{1+i} \\
\Rightarrow \text{YTM, } i = \frac{110 - 100}{100} = 0.10 = 10\%
\]

**Note:** Simple loan \( \Rightarrow \) simple interest = YTM

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### Types of Credit Instruments

**Simple loan**
[Principal + simple Interest] paid to lender at given maturity date.

**Fixed-payment loan**
fixed payment (incorporating part of the principal and interest payment) paid over a period of time. Ex: car loan

**Coupon bond**
Pays owner of bond a fixed (coupon) payment, until maturity when it pays off face (par) value. Ex: corporate bond (capital market instrument)

**Discount (zero coupon) bond**
Bought at price below face value (i.e., at a discounted), and face value repaid at maturity. Ex: US treasury bill (money market instrument)

- Distinguish: simple Interest rate, coupon rate, fixed payment rate, yield to maturity
- We’ll calculate the Yield to Maturity in each case.
Equations and Unknowns

- **1 equation, 1 unknown**
  \[ 4x = x + 2 \quad \Rightarrow \quad x = \frac{2}{3} \]

- **2 equations, 2 unknowns**
  \[
  \begin{align*}
  &y = 4 - x \\
  &y = 1 + x
  \end{align*}
  \quad \Rightarrow \quad \begin{cases} x = \frac{3}{2}, & y = \frac{5}{2} \end{cases}
  \]

- **1 equation, 2 unknowns**
  \[ y = x + 1 \quad \Rightarrow \quad \text{different } y \text{ for different } x \]
  \[ \{x = 1, \ y = 2\}, \ {x = 0.5, \ y = 1.5\}, \ {x = -2, \ y = -1\} \]

Yield to Maturity: Discount Bonds

- **Example**
  \[ P = \$900, \ F = \$1000, \ \text{one year} \]
  \[
  $900 = \frac{\$1000}{1+i}
  \]
  \[ \Rightarrow i = \frac{\$1000 - \$900}{\$900} = 0.111 = 11.1\% \]

- **Generalizing,**
  \[ i = \frac{F - P}{P} \]
Yield to Maturity (YTM): Fixed Payment Loans

- **Example:** Loan value (LV) = $1000
  Make fixed payment (FP) = $126, every year, for 25 years

\[
1000 = \frac{126}{1+i} + \frac{126}{(1+i)^2} + \frac{126}{(1+i)^3} + \ldots + \frac{126}{(1+i)^{25}}
\]

Solve the equation to obtain YTM, \( i = 12\% \)
If, FP = $85.81, then YTM = 7%

- **Generalizing for \( n \)-years,**

\[
LV = \frac{FP}{1+i} + \frac{FP}{(1+i)^2} + \frac{FP}{(1+i)^3} + \ldots + \frac{FP}{(1+i)^n}
\]

Yield to Maturity: Coupon Bonds

- **Example:** Face value (\( F \)) = $1000,
  Coupon payment (\( C \)) = $100, (i.e., Coupon rate = 10% = \( C/F \))
  (Current) price of bond = \( P \)

\[
P = \frac{100}{(1+i)} + \frac{100}{(1+i)^2} + \frac{100}{(1+i)^3} + \ldots + \frac{100}{(1+i)^{10}} + \frac{1000}{(1+i)^{10}}
\]

- **Generalizing for \( n \)-years**

\[
P = \frac{C}{(1+i)} + \frac{C}{(1+i)^2} + \frac{C}{(1+i)^3} + \ldots + \frac{C}{(1+i)^n} + \frac{F}{(1+i)^n}
\]
Yield to Maturity: Coupon Bonds

- Face value \((F) = $1000\),
- Coupon payment \((C) = $100\), (Current) price of bond = \(P\)

\[
P = \frac{100}{1+i} + \frac{100}{(1+i)^2} + \frac{100}{(1+i)^3} + \ldots + \frac{100}{(1+i)^{10}} + \frac{1000}{(1+i)^{10}}
\]

**Notes:**
(a) At the end period you have to account for both the Coupon payment and the face value.
(b) Both \(P\) and \(i\) are unknowns in these equations.
(c) What is the relation between \(P\) and \(i\)? Once you know one you also know the other.

(d) How does any one of these get determined? DD and SS mechanism gives one of these.

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Coupon Bonds: Price and YTM

1. When bond is at par (i.e. \(P=F\)), yield equals coupon rate
   Compare with depositing $1000 in a bank account with 10\% (non-compound) interest.
2. Price and yield-to-maturity \((i)\) are negatively related
   As \(i\) increase, the denominators increase \(\Rightarrow\) each RHS term decreases.
3. If, Yield > (coupon rate), bond price is below par value
   Start with \(P=F\). So, from (1), we know that \(i = \text{coupon rate}\). Now, suppose if \(i\) increases, then from (2), we know that \(P\) must decreases.

<table>
<thead>
<tr>
<th>Price of Bond ($)</th>
<th>Yield to Maturity (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,200</td>
<td>7.13</td>
</tr>
<tr>
<td>1,100</td>
<td>7.48</td>
</tr>
<tr>
<td>1,000</td>
<td>10.00</td>
</tr>
<tr>
<td>900</td>
<td>11.75</td>
</tr>
<tr>
<td>800</td>
<td>13.81</td>
</tr>
</tbody>
</table>
Perpetuity: A Special Coupon Bond

- Also known as a ‘Consol’
- $C =$ yearly payment, $P_c =$ price, $i_c =$ YTM
- Formula for YTM
  \[
P_c = \frac{C}{1+i_c} + \frac{C}{(1+i_c)^2} + \frac{C}{(1+i_c)^3} + \ldots
  \]
  \[
  = \frac{C}{i_c}
  \]
  \[
  \Rightarrow \quad i_c = \frac{C}{P_c}
  \]

Perpetuity: Derivation

Formula for an infinite sum:

\[
1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}, \quad \text{for } x < 1.
\]

Now, consider the price of the consol,

\[
P_c = \frac{C}{1+i_c} + \frac{C}{(1+i_c)^2} + \frac{C}{(1+i_c)^3} + \ldots = C\left[\frac{1}{1+i_c} + \frac{1}{(1+i_c)^2} + \frac{1}{(1+i_c)^3} + \ldots\right]
\]

\[
= C\left[-1+1+\left(\frac{1}{1+i_c}\right) + \left(\frac{1}{1+i_c}\right)^2 + \left(\frac{1}{1+i_c}\right)^3 + \ldots\right].
\]

First, let $\frac{1}{1+i_c} = x$. Note that, $i_c > 0 \Rightarrow 1+i_c > 1 \Rightarrow \frac{1}{1+i_c} < 1$

so that

\[
\left[\text{ex: if } i_c = 0.10, \quad 1+0.10 = 1.10 > 1, \quad \frac{1}{1.10} = 0.91 < 1\right].
\]

Therefore,

\[
P_c = C\left[-1+\frac{1+i_c}{i_c}\right] = C\left[\frac{1+i_c}{i_c}\right] = \frac{C}{i_c}.
\]
Perpetuity, continued . . .

\[ P_c = \frac{C}{i_c} \implies i_c = \frac{C}{P_c} \]

- The inverse relationship between Price and YTM is straight-forward
- Note: \( i_c \) is also known as “Current yield”
- Consols are rare. But we can use the formula to approximate calculations in coupon bonds

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Approximate Measure for YTM of Coupon Bond: Current Yield

- Difficult to calculate \( i (= \text{YTM}) \) of a coupon bond from
  \[ P = \frac{C}{(1+i)} + \frac{C}{(1+i)^2} + \ldots + \frac{C}{(1+i)^n} + \frac{F}{(1+i)^n} \]
- But, when a coupon bond has long maturity, it is ‘like’ a consol and \( i_c \) is a good approximation of \( i (= \text{YTM}) \) when maturity period is long
- So, instead calculate, \( i_c = \frac{C}{P} \)

Further Justifications

- When \( P=F \), YTM = coupon rate.
  Notice that when \( P=F \), \( i_c \), coupon rate.
- Regardless of whether \( i_c \) is a good approximation or not, a change in \( i_c \) always signals a change in YTM in the same direction [property of inverse relation between P and YTM].
YTM/Interest Rate ($i$) versus Rate of Return ($R$)

- Compare two scenarios of a coupon bond
  - (a) Next period is not the maturity period but you sell it next period: get $C$ for one period and selling price
  - (b) Next period is the maturity period: get $C$ and $F$

**Case (a)**

$$R = \frac{C + P_{t+1} - P_t}{P_t} = \frac{C}{P_t} + \frac{P_{t+1} - P_t}{P_t} = \frac{i_c}{g}$$

∴ $R = i_c + g = \text{current yield + capital gain}$

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YTM/Interest Rate ($i$) versus Rate of Return ($R$)

- Case (b) Next period is the maturity period, then

$$P_t = \frac{C}{1+i} + \frac{F}{1+i} \quad \Rightarrow \quad i = \frac{C + F - P_t}{P_t}$$

$$\Rightarrow \quad i = \frac{C}{P_t} + \frac{F - P_t}{P_t}$$

- If, Holding period = maturity period, $R = \text{YTM}$
- If, Holding period < maturity period, $R = i + g$
You have bought the bond at $P_t$, now Interest rate ↑

**Case(a): holding < maturity**

$$R = \frac{C}{P_t} + \frac{P_{t+1} - P_t}{P_t} = i_c + g$$

→ current yield does not change
→ $P_{t+1}$ will decrease
→ capital gain decreases

**Case(b): holding period = maturity**

$$R = \frac{C}{P_t} + \frac{F - P_t}{P_t}$$

→ $F$ not affected by changes in interest rate
→ $R$ remains unchanged

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What happens if the bond matures next year ($t+1$)?

Then change in interest rate has no effect because $F$ is unaffected by interest rate change ($R$=today’s $i$)

“Interest rate risk” → If holding period is less than maturity period then, a rise in $i$ lowers $R$.

Prices and returns more volatile for long-term bonds because they have higher interest-rate risk
Only bond whose return = yield is one where, maturity = holding period

For bonds with: holding period < maturity period
\[ i \uparrow \quad P \downarrow \quad \text{implying capital loss}\n
<table>
<thead>
<tr>
<th>Years to Maturity When Bond Is Purchased</th>
<th>Initial Yield (%)</th>
<th>Initial Price ($)</th>
<th>Current Price ($)</th>
<th>Next Year* Price ($)</th>
<th>Rate of Capital Gain (%)</th>
<th>Rate of Return (2 + 5) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>10</td>
<td>1,000</td>
<td>503</td>
<td>-49.7</td>
<td>-39.7</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>1,000</td>
<td>516</td>
<td>-48.4</td>
<td>-38.4</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>1,000</td>
<td>597</td>
<td>-40.3</td>
<td>-30.3</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1,000</td>
<td>741</td>
<td>-25.9</td>
<td>-13.9</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1,000</td>
<td>917</td>
<td>-8.3</td>
<td>+1.7</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>1,000</td>
<td>1,000</td>
<td>0.0</td>
<td>+10.0</td>
<td></td>
</tr>
</tbody>
</table>

Longer is maturity, greater is % price change associated with interest rate change

Longer is maturity, more return changes with change in interest rate

Bond with high initial interest rate can still have negative return if \[ i \uparrow \]
Distinction Between Real and Nominal Interest Rates

Real Interest rate is adjusted for expected changes in the price level

\[ i_r = i - \pi^e \]

[ex: if \( i = 5\% \) and \( \pi^e = 3\% \) then: \( i_r = 5\% - 3\% = 2\% \)]

- Real interest rate more accurately reflects true cost of borrowing
- When real rate is low, greater incentives to borrow and less to lend
- \( i_r \) can be negative [ex: \( i = 8\% \), \( \pi^e = 10\% \)]
- Similar concept: Real Return

U.S. Real & Nominal Interest Rates